Problem Set 4

Question 1 (Single-Peaked Preferences). Consider the social choice problem in which all agents have *single-peaked preferences* over *two dimensional* allocation spaces. Formally, the allocation space is the unit square $A = [0,1] \times [0,1] \subseteq \mathbb{R}^2$. An outcome is a single point $\mathbf{x} \in A$. Each agent i has a weak preference ordering $\succeq_i$ over the outcomes in A. We assume the preference relation $\succeq_i$ is *single-peaked*: there exists a point $\mathbf{p}_i = (x_i, y_i) \in A$ for each agent i such that for all $\mathbf{x} \in A \setminus \{p_i\}$ and all $\lambda \in [0,1)$, $(\lambda\mathbf{x} + (1-\lambda)\mathbf{p}_i) \succ_i \mathbf{x}$. That is, under a single-peaked preference relation, preference is *strictly* decreasing as one moves away from $\mathbf{p}_i$. The social choice function f takes agents' preference $(\succeq_1, \ldots, \succeq_n)$ as input, and output an outcome $\mathbf{x} \in A$.

1. Prove that the social choice function outputting the average of the peaks

   $$f(\succeq_1, \ldots, \succeq_n) = \frac{1}{n}\sum_{i=1}^{n}\mathbf{p}_i = \left(\frac{1}{n}\sum_{i=1}^{n}x_i, \frac{1}{n}\sum_{i=1}^{n}y_i\right)$$

   is not strategy-proof.

2. Suppose n is an odd number. Prove that the social choice function outputting the median of the x-coordinates of the peaks

   $$f(\succeq_1, \ldots, \succeq_n) = \mathbf{p}_i = (x_i, y_i), \quad \text{where } x_i \text{ is the median of } \{x_1, \ldots, x_n\}$$

   is not strategy-proof.

3. Suppose n is an odd number. Prove that the social choice function outputting the median of both coordinates of the peaks

   $$f(\succeq_1, \ldots, \succeq_n) = (x_i, y_j), \quad \text{where } \begin{array}{l} x_i \text{ is the median of } \{x_1, \ldots, x_n\} \\ y_j \text{ is the median of } \{y_1, \ldots, y_n\} \end{array}$$

   is not strategy-proof.

4. For $\mathbf{a}_1 = (x_1, y_1), \mathbf{a}_2 = (x_2, y_2) \in A$, let $d(\mathbf{a}_1, \mathbf{a}_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ be the distance between $\mathbf{a}_1$ and $\mathbf{a}_2$. Consider the case where each $\succeq_i$ satisfies the following additional property: if $\mathbf{a}_1, \mathbf{a}_2$ satisfies $d(\mathbf{a}_1, \mathbf{p}_i) = d(\mathbf{a}_2, \mathbf{p}_i)$, then $\mathbf{a}_1 \succeq_i \mathbf{a}_2$ and $\mathbf{a}_2 \succeq \mathbf{a}_1$. That is, each agent equally prefers any two points that have equal distance to his peak.

   Suppose n is an odd number. Prove that the social choice function in Part 3 is strategy-proof.

Question 2 (House Allocation). 1. Consider the house allocation problem with *strict* preferences. Complete the proof of Theorem 10.6 in the book by proving that the allocation output by the Top Trading Cycle Algorithm does not contain a blocking coalition of agents.

2. Is the allocation output by the Top Trading Cycle Algorithm Pareto-optimal?

3. Consider the setting where the preferences may not be strict. Now each agent may have more than one outgoing edge. Suppose the Top Trading Cycle Algorithm iteratively finds an *arbitrary* cycle and swaps the houses according to the cycle. Is the output allocation Pareto-optimal?

Question 3 (Stable Matching). Consider the stable matching problem with a set M of males and a set N of females, with $|M| = |N| = T$. Each male $m \in M$ has a *valuation function* $f_m : N \to \mathbb{R}^+$, which yields a ranking over the females. The ranking is assumed to be strict: for any $n_1, n_2 \in N$, we have $f_m(n_1) \neq f_m(n_2)$. Each female $n \in N$ has a *valuation function* $f_n : M \to \mathbb{R}^+$, which yields a ranking that is also assumed to be strict.

1. Consider the perfect matching μ that maximizes the social welfare: $\sum_{m=1}^{T} f_m(\mu(m)) + \sum_{n=1}^{T} f_n(\mu(n))$. Is μ always stable?

2. Consider a stable matching μ. Is μ Pareto-optimal (meaning that there does not exist μ', which may or may not be stable, such that every agent in $M \cup N$ receives weakly higher value and at least one agent receives strictly higher value)?

3. Consider a male-optimal matching μ. Prove that μ is *weakly* Pareto-optimal for males (i.e., there does not exist μ', which may or may not be stable, such that every *male* receives *strictly* higher value).

4. Consider a male-optimal matching μ. Prove that μ may not be Pareto-optimal for males (i.e., for some male-optimal μ, there may exist μ', which may or may not be stable, such that every male receives weakly higher value and at least one male receives strictly higher value).